Solution Of Linear And Nonlinear Partial Differential Equations Using Mixture Of Iman Transform And The Projected Differential Transform Method

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Abstract

The aim of this study is to solve some linear and nonlinear partial differential equations using the new integral transform "Iman transform" and projected differential transform method. The nonlinear terms can be handled by using of projected differential transform method; this method is more efficient and easy to handle such partial differential equations in comparison to other methods. The results show the efficiency and validation of this method.

Keywords: Iman transform, projected differential transform method, nonlinear partial differential equations.

1. Introduction

Many problems of physical interest are described by linear and nonlinear partial differential equations with initial or boundary conditions, these problems are fundamental importance in science and technology especially in engineering. Some valuable contributions have already been made to showing differential equations such as Laplace transform method [lslam, Yasir Khan, Naeem Faraz and Francis Austin (2010), Lokenath Debnath and Bhatta (2006)], Sumudu transform [A.Kilicman and H.E.Gadain (2009), (2010)], differential transform method etc. Iman transform method is very effective tool for solve linear partial

differential equations .In this study we use the projected differential

transform method [Nuran Guzel and Muhammet Nurulay (2008), Shin- Hsiang Chang , Ling Chang (2008)] to decompose the nonlinear terms, this means that we can use both Iman transform and projected differential transform methods to solve many nonlinear partial differential equations.

1.1.   *Iman Transform:*

1.2.   The basic definitions of modified of Sumudu transform or Iman transform is defined as follows, Iman transform of the function $f(t)$ is

$$I[f(t)] = \frac{1}{v^2} \int_0^\infty f(t)e^{-tv^2} dt = L\ (v),\ t \geq 0$$

In this paper, we combined Iman transform and projected differential transform methods to solve linear and nonlinear partial differential equations.

To obtain Iman transform of partial derivative we use integration by parts, and then we have:

$$I\left[\frac{\partial f(x,t)}{\partial t}\right] = v^2 L(x,v) - \frac{1}{v^2} f(x,0) I\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{d}{dx}[L(x,v)]$$

$$I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = v^4 L(x,v) - f(x,0) - \frac{1}{v^2}\frac{\partial f(x,0)}{\partial t} I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{d^2}{dx^2}[L(x,v)]$$

Proof:

To obtain Iman transform of partial derivatives we use integration by parts as follows:

$$I\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^\infty \frac{1}{v^2}\frac{\partial f}{\partial t}e^{-tv^2} dt = \lim_{p\to\infty} \int_0^p \frac{1}{v^2}e^{-tv^2}\frac{\partial f}{\partial t} dt = \lim_{p\to\infty}\left\{\left[\frac{1}{v^2}e^{-tv^2}f(x,t)\right]_0^p - \int_0^\infty f(x,t)e^{-tv^2} dt\right\}$$

$$= v^2 L(x,v) - \frac{1}{v^2}f(x,0)$$

We assume that f is piecewise continuous and it is of exponential order.

Now

$$I\left[\frac{\partial f}{\partial x}\right] = \int_0^\infty \frac{1}{v^2}e^{-tv^2}\frac{\partial f(x,t)}{\partial x} dt = \frac{\partial}{\partial x}\int_0^\infty \frac{1}{v^2}e^{-tv^2}f(x,t)\ dt$$

using the Leibnitz rule to find:

$$I\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}[L(x,v)]$$

By the same method we find

$$I\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[L(x,v)]$$

To find

$$I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]$$

Let $\frac{\partial f}{\partial t} = g$,Then we have

$$I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = I\left[\frac{\partial g(x,t)}{\partial t}\right] = v^2 g(x,t) - \frac{1}{v^2}g(x,0)$$

$$I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = v^4 L(x,v) - f(x,0) - \frac{1}{v^2}\frac{\partial f(x,0)}{\partial t}$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

1.2. Projected Differential Transform Methods:

In this section we introduce the projected differential transform method which is modified of the differential transform method.

Definition:

The basic definition of projected differential transform method of function $f(x_1, x_2, \ldots, x_n)$ is defined as

$$f(x_1, x_2, \ldots, x_{n-1}, k) = \frac{1}{k!}\left[\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_n^k}\right]_{x_n=0} \qquad (1)$$

Such that $f(x_1, x_2, \ldots, x_n)$ is the original function and $f(x_1, x_2, \ldots, x_{n-1}, k)$ is projected transform function.

And the differential inverse transform of $f(x_1, x_2, \ldots, x_{n-1}, k)$ is defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty} f(x_1, x_2, \ldots, x_n)(x - x_0)^k \qquad (2)$$

The fundamental theorems of the projected differential transform are

Theorems:

$$(1) If \ z(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n) \pm v(x_1, x_2, \ldots, x_n)$$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = u(x_1, x_2, \ldots, x_{n-1}, k) \pm (x_1, x_2, \ldots, x_{n-1}, k)$$

$(2) If \ z(x_1, x_2, \ldots, x_n) = c \, u(x_1, x_2, \ldots, x_n)$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = c \, u(x_1, x_2, \ldots, x_{n-1}, k)$$

$(3) If \ z(x_1, x_2, \ldots, x_n) = \dfrac{du(x_1, x_2, \ldots, x_n)}{dx_n}$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = (k+1) \, u(x_1, x_2, \ldots, x_{n-1}, k+1)$$

$(4) If \ z(x_1, x_2, \ldots, x_n) = \dfrac{d^n u(x_1, x_2, \ldots, x_n)}{dx_n^n}$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = \frac{(k+n)!}{k!} u(x_1, x_2, \ldots, x_{n-1}, k+n)$$

$$(5) If \ z(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n) \, v(x_1, x_2, \ldots, x_n)$$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = \sum_{m=0}^{k} u(x_1, x_2, \ldots, x_{n-1}, m) \, v(x_1, x_2, \ldots, x_{n-1}, k-m)$$

$$(6) If \ z(x_1, x_2, \ldots, x_n) = \quad u_1(x_1, x_2, \ldots, x_n) u_2(x_1, x_2, \ldots, x_n) \ldots u_n(x_1, x_2, \ldots, x_n)$$

Then

$$z(x_1, x_2, \ldots, x_n) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} u_1(x_1, x_2, \ldots, x_{n-1}, k_1) u_2(x_1, x_2, \ldots, x_{n-1}, k_2-k_1)$$

$$\text{X} \ldots\ldots u_{n-1}(x_1, x_2, \ldots, x_{n-1}, k_{n-1}-k_{n-2}) u_n(x_1, x_2, \ldots, x_{n-1}, k-k_{n-1})$$

$$(7) If \ z(x_1, x_2, \ldots, x_n) = x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}$$

Then

$$z(x_1, x_2, \ldots, x_{n-1}, k) = \delta(x_1, x_2, \ldots, x_{n-1}, q_n - k) = \begin{cases} 1 & q_n = k \\ 0 & q_n \neq k \end{cases}$$

Note that c is a constant and n is a nonnegative integer.

Table of Functions and their Iman Transform

$f(t)$	$I[f(t)] = F(v)$
1	$\dfrac{1}{v^4}$
t	$\dfrac{1}{v^6}$
t^2	$\dfrac{2!}{v^8}$
$t^n \ n \in N$	$\dfrac{n!}{v^{2n+4}}$
e^{at}	$\dfrac{1}{v^2(v^2 - a)}$
$\sin(at)$	$\dfrac{a}{v^2(v^4 + a^2)}$
$\cos(at)$	$\dfrac{1}{v^4 + a^2}$

2. Applications:

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{3}$$

$$u(x,0) = h(x), u_t(x,0) = f(x)$$

Where D is linear differential operator of order two, R is linear differential operator of less order than D, N is the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking Iman transform on both sides of equation (3), to get:

$$I[Du(x,t)] \square\square I[Ru(x,t)] \square\square I[Nu(x,t)] \ \square I[g(x,t)] \tag{4}$$

Using the differentiation property of Iman transforms and above initial conditions, we have:

$$I[u(x,t)] \ \square \frac{1}{v^4} I[g(x,t)] \square \frac{1}{v^4} h(x) \ \square \frac{1}{v^6} f(x) \ \square \frac{1}{v^4} I[Ru(x,t) \ \square\square Nu(x,t)] \tag{5}$$

Appling the inverse Iman transform on both sides of equation (5), to find:

$u \quad (x \quad ,t \quad) \quad \Box\Box G(x \quad ,t \quad) \quad \Box\Box I^{-1}\Box\frac{1}{v^4}I \quad \Box Ru \quad (x \quad ,t \quad) \quad \Box\Box Nu \quad (x \quad ,t$

$)\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box(6)$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the projected differential transform method.

$u (x , \text{m+1}) \Box\Box\Box\Box I^{-1}\Box\frac{1}{v^4}I \Box A_m + B_m\Box\Box\Box\Box\Box\Box u(x,0) = G(x,t)(7)$

Where A_m ,B_m are the projected differential transform of $Ru (x ,t)$, $Nu (x ,t)$.

From equation (7), we have

$u(x,0) = G(x,t) \quad , u(x,1) =\Box\Box\Box\Box I^{-1}\Box\frac{1}{v^4}I \Box A_0 + B_0\Box\Box\Box$

$u(x,2) =\Box\Box\Box\Box I^{-1}\Box\frac{1}{v^4}I \Box A_1 + B_1\Box\Box\Box \quad , u(x,3) =\Box\Box\Box\Box I^{-1}\Box\frac{1}{v^4}I \Box A_2 + B_2\Box\Box\Box$

…………..

Then the solution of equation (3) is

$u (x ,t) \Box u (x ,0) \Box u (x ,1) \Box u (x ,2) \Box ....$

To illustrate the capability and simplicity of the method, some examples for different linear and nonlinear partial differential equations will be discussed.

Example 2.1:

Consider the simple first order partial differential equation

$$\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y \quad , y(x,0) = 6e^{-3x} \quad\quad (8)$$

Taking Iman transform of (8), leads to

$$I[y(x,t)] = \frac{6e^{-3x}}{v^4} + \frac{1}{2v^2}I[A_m - B_m]$$

Take the inverse Iman transform to find

$$y(x,m+1) = I^{-1}\left\{\frac{1}{2v^2}I[A_m - B_m]\right\}, y(x,0) = 6e^{-3x} \quad\quad (9)$$

Where

$A_m = \frac{\partial y(x,m)}{\partial x} \quad , B_m = y(x,m)$

are the projected differential transform of$\frac{\partial y(x,t)}{\partial x} \quad , y(x,t)$

The standard Iman transform defines the solution $y (x ,t)$ by the series

$y(x,t) = \sum_{m=0}^{\infty} y(x,m)$

From equation (9) we find that: $y(x,0) \square\square 6e^{-3x}$

$A_0 = -18e^{-3x}, \quad B_0 \square\square 6e^{-3x} \square\square y(x,1) = I^{-1}\left\{\frac{-12e^{-3x}}{v^6}\right\} = -12te^{-3x}$

$A_1 = 36te^{-3x}, \quad B_1 \square\square -12e^{-3x}, y(x,2) = I^{-1}\left\{\frac{24e^{-3x}}{v^8}\right\} = 12t^2 e^{-3x}$

$$y(x,3) == -8t^3 e^{-3x}$$

The solution in a series form is given by

$y(x,t) = 6e^{-3x} - 12te^{-3x} + 12t^2 e^{-3x} + \cdots = 6e^{-3x}e^{-2t} = 6e^{-(3x+2t)}$

Example 2.2:
Consider the following linear second order partial differential equation
$$u_{xx} + u_{tt} = 0, u(x,0) = 0, u_t(x,0) = \cos x \qquad (10)$$

To find the solution we take Iman transform of equation (10) and making use of the conditions to find,
$$I[u(x,t)] = \frac{\cos x}{v^6} - \frac{1}{v^4} I[A_m] A_m = \frac{\partial^2 u(x,m)}{\partial x^2}$$

Take the inverse Iman transform we get:
$$u(x,m+1) = -I^{-1}\left\{\frac{1}{v^4} I[A_m]\right\}, u(x,0) = 0, u(x,1) = t \cos x \qquad (11)$$

By using equation (11), we find that:

$A_1 = -t \cos x, \quad , u(x,2) = I^{-1}\left\{\frac{1}{v^4} I[t \cos x]\right\} = \frac{t^3}{3!} \cos x$

$A_1 = \frac{t^3}{3!} \cos x, \quad , u(x,3) = I^{-1}\left\{\frac{1}{v^4} I\left[\frac{t^3}{3!} \cos x\right]\right\} = \frac{t^5}{5!} \cos x$

.
.
.

$$\text{Then the solution is } u(x,t) = \cos x \left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right] = \cos x \, \sinh t$$

3. Conclusion
The solution of linear and nonlinear partial differential equations can be obtained using Iman transform and projected differential transform method without any discretization of the variables. The results for all examples can be obtained in

series form, and all calculations in the method are very easy. This technique is
useful to solve linear and nonlinear partial differential equations.

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